Addendum to “Improving the Two-Pass Resampling Algorithm”

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Proofs of Mathematical Statements

Statement 1: Let $x(u,v) = \frac{au+bv+d}{mu+nv+p}$ where $mu+nv+p \neq 0$ for all $0 \leq u,v \leq 1$. If the sign of $\left|\frac{\partial x}{\partial u}\right|$ at $v=0$ is different from its sign at $v=1$ then $\frac{\partial y/\partial u}{\partial x/\partial u} = \frac{(en-mf)v+ep-mh}{(an-mb)v+ap-md}$ is unbounded in the inverval $[0,1]$. Otherwise the maximum of the ratio in this interval is attained at $v=0$ or $v=1$.

Proof: Since its derivative is never zero, the ratio is either constant or strictly monotonic. If it is constant then the maximum is attained everywhere. Otherwise, it is strictly monotonic. If the function is continuous in the interval then its maximum is attained at 0 or 1. The ratio fails to be continuous only where the denominator vanishes; but the denominator is $\frac{\partial x}{\partial u} = \frac{(an-bm)v+ap-dm}{(mu+nv+p)^2}$, and it vanishes only where $(an-bm)v+ap-dm=0$. If $(an-bm)v+ap-dm=0$ everywhere then the maximum is attained everywhere. Otherwise $(mu+nv+p)^2$ is positive and then $(an-bm)v+ap-dm$ must change sign where it vanishes, and that can happen only once. If that happens between 0 and 1, then the sign of $\frac{\partial y/\partial u}{\partial x/\partial u}$ must differ at 0 and 1.

Statement 2: The bounds of the partial derivatives of a projective transformation with a non-vanishing denominator over the unit square $\{0 \leq u,v \leq 1\}$ are attained at the corners $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

Proof: It suffices to prove the statement for $f(u,v) = \frac{\partial x}{\partial v} = \frac{(bm-an)u+bp-dn}{(mu+nv+p)^2}$, the proofs for the other partial derivatives are similar. The denominator doesn’t vanish in the unit square, hence f is differentiable there, it has a maximum, and a maximum in the interior of the square would imply $\frac{\partial f}{\partial v} = 0$ and $\frac{\partial f}{\partial u} = 0$ simulataneously. Write $f(u,v) = \frac{qu+r}{(mu+nv+p)^2}$ with $q = bm-an$ and $r = bp-dn$. Then $\frac{\partial f}{\partial v} = \frac{-2n(qu+r)}{(mu+nv+p)^3}$, and it vanishes only if either $n=0$ or $qu+r=0$. If n=0 then $f(u,v) = \frac{bmu+bp}{(mu+p)^2} = \frac{b}{mu+p}$. In that case, $\frac{\partial f}{\partial u} = \frac{-bm}{(mu+p)^2} = 0$ implies either $b=0$, and then $f(u,v)=0$ everywhere, or $m=0$, and then $f(u,v) = \frac{b}{p}$ everywhere. In either case, f is constant, attaining its maximum everywhere. If $qu+r=0$, then $\frac{\partial f}{\partial u} = \frac{q(mu+nv+p)^2-2m(mu+nv+p)(qu+r)}{(mu+nv+p)^4} = \frac{q}{(mu+nv+p)^2}$, and $\frac{\partial f}{\partial u} = 0$ implies $q=0$, hence $f(u,v) = \frac{r}{(mu+nv+p)^2}$. If $r=0$ then $f=0$ everywhere, attaining its maximum everywhere. Otherwise f has a local maximum only where the denominator has a local minimum, and that happens only where it vanishes. But the denominator does not vanish in the unit square, hence a local maximum is impossible, and the maximum is attained along the boundary. Again, on a $u=c$ edge ($u=0$ or $u=1$) the function $g(v) = \frac{r}{(nv+(mc+p))^2}$ has no local maximum between 0 and 1, because its denominator is positive there; a similar argument rules out an interior maximum on any constant v edge. The maximum must therefore be attained at a corner.

Pseudocode

The following procedure uses the aliasing and bottleneck errors to select a two pass decomposition for a given projective matrix. The only input to the procedure is the transformation matrix with elements (a, b, d, e, f, h, m, n, p), see equation 11, and it outputs the optimal decomposition.

$\textsc{SelectDecomposition}(a,b,d,e,f,h,m,n,p : REAL)$

comment: Compute $max\left|\frac{\partial x}{\partial u}\right|$ using Eq. 18.

comment: Similarly compute the maxima of the three remaining partial derivatives.

$error \leftarrow max\left|\frac{\partial y}{\partial v}\right| * max\left|\frac{\partial x}{\partial v}\right| + ComputeBottleneckError(a,b,d,e,f,h,m,n,p,RowsFirst)$

$selection \leftarrow RowsFirstNoRotation$

if $(max\left|\frac{\partial x}{\partial u}\right| * max\left|\frac{\partial y}{\partial u}\right| + ComputeBottleneckError(a,b,d,e,f,h,m,n,p,ColsFirst) < error)$

then $\begin{cases} error \leftarrow max\left|\frac{\partial x}{\partial u}\right| * max\left|\frac{\partial y}{\partial u}\right| + ComputeBottleneckError(a,b,d,e,f,h,m,n,p,ColsFirst) \\ selection \leftarrow ColsFirstNoRotation \end{cases}$

if $(max\left|\frac{\partial y}{\partial u}\right| * max\left|\frac{\partial x}{\partial u}\right| + ComputeBottleneckError(b,-a,d,f,-e,h,n,-m,p,RowsFirst) < error)$

then $\begin{cases} error \leftarrow max\left|\frac{\partial y}{\partial u}\right| * max\left|\frac{\partial x}{\partial u}\right| + ComputeBottleneckError(b,-a,d,f,-e,h,n,-m,p,RowsFirst) \\ selection \leftarrow RowsFirstPreRotation \end{cases}$

if $(max\left|\frac{\partial x}{\partial v}\right| * max\left|\frac{\partial y}{\partial v}\right| + ComputeBottleneckError(b,-a,d,f,-e,h,n,-m,p,ColsFirst) < error)$

then $\begin{cases} error \leftarrow max\left|\frac{\partial x}{\partial v}\right| * max\left|\frac{\partial y}{\partial v}\right| + ComputeBottleneckError(b,-a,d,f,-e,h,n,-m,p,RowsFirst) \\ selection \leftarrow ColsFirstPreRotation \end{cases}$

return $(selection)$

COMPUTEBOTTLENECKERROR($a,b,d,e,f,h,m,n,p : REAL, order : SCANORDER$)

if $(order == RowsFirst)$

then
- **if** $((fp-hn)(fm-en+fp-hn) <= 0)$
- **then return** (∞)
- **else**
  - **comment:** Compute maximum of the fraction of partial derivatives according to Eq. 13.
  - **return** $\left(max\left\{\left|\frac{\partial y/\partial u}{\partial x/\partial u}\right|\right\}\right)$

else if $((order == ColsFirst))$

then
- **if** $((bm-na)(fn-ne+fp-nh) <= 0)$
- **then return** (∞)
- **else**
  - **comment:** Compute maximum of the fraction of partial derivatives according to Eq. 15.
  - **return** $\left(max\left\{\left|\frac{\partial x/\partial v}{\partial y/\partial v}\right|\right\}\right)$